On the quantization of angular momentum

1994 J. Phys. A: Math. Gen. 27 6237

(http://iopscience.iop.org/0305-4470/27/18/031)

J. Phys. A: Math. Gen. 27 (1994) 6237–6241. Printed in the UK

On the quantization of angular momentum

Vu B Ho
Department of Physics, Monash University, Clayton, Victoria 3168, Australia

Received 7 December 1993, in final form 7 June 1994

Abstract. It is shown that when a hydrogen-like atom is treated as a two-dimensional system whose configuration space is multiply connected, then, in order to obtain the same energy spectrum as in the Bohr model, the angular momentum must be half-integral.

In quantum mechanics, there are two alternative methods for quantization of a physical system: the canonical quantization and the Feynman path-integral formulation. While the more familiar canonical quantization replaces classical observables by operators which obey Heisenberg commutation relations and, hence, the mathematics one invokes is that of operators in Hilbert space, the path-integral formulation of quantum mechanics is based on the concept of the transition amplitude to which all possible paths contribute [1–4]. However, although the Feynman formulation is closely related to the canonical equations of quantum mechanics, there remain some profound differences between the two methods concerning the nature of physical observables. With the method of canonical quantization, due to the Heisenberg uncertainty relation, it is not possible to define a path in the classical sense for a particle to move from one place to another in space. On the other hand, when the Feynman path-integral method of quantization is employed, then, since this method does not exclude the notion of classical paths of a particle, it is allowable to discuss the dynamics of the particle in a classical sense, until the complex-valued transition amplitude is introduced to describe the quantum dynamics of the particle. In particular, we are allowed to consider all possible paths in different frames of reference in different configuration spaces. In this paper, we will discuss this possibility and show that the topology of a configuration space can determine the quantization of the angular momentum.

The problem we will consider is that of coordinate transformations in quantum mechanics and the quantization of angular momentum in multiply-connected spaces. In quantum mechanics, the problem of multivalued angular-momentum eigenfunctions may be argued to appear only because one changes from cartesian coordinates to polar coordinates which are singular at the coordinate origin [5]. On the other hand, recent developments in quantum mechanics in multiply-connected spaces have shown that, within the present physical interpretation of quantum mechanics, the use of multivalued wavefunctions is allowable provided the configuration space is non-simply connected [6, 7]. In the following, we will discuss the particular case of a hydrogen-like atom from the point of view of an observer who is in a coordinate system that describes the atom as a planar physical system. In this case, if we assume that the electron can never be able to penetrate the nucleus, then the configuration space can be considered as multiply connected and the use of multivalued wavefunctions is allowed. In such a situation, the single-valuedness condition is no longer obviously a good requirement for the angular momentum to be integral. However, we will

0305-4470/94/186237+05$19.50

show that with this observer, the eigenvalues of angular momentum must be half-integral if the observer obtains the same spectrum of energy as that of an observer who observes the atom in three-dimensional space.

The hydrogen-like atom consisting of a single electron of charge $-e$ and a nucleus of charge Ze is described by the eigenvalue equation

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(\boldsymbol{r}) - \frac{Ze^2}{r}\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \tag{1}$$

where μ is the reduced mass. We will consider the case of bound states with $E < 0$. In three-dimensional space, when the wavefunction is written in the form

$$\psi(\boldsymbol{r}) = R(r)Y_{l,m}(\theta, \phi) \tag{2}$$

we get the radial equation for the function $R(r)$ [8, 9]

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\rho^2} + \frac{2}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho} - \frac{l(l+1)}{\rho^2}R + \frac{\lambda}{\rho}R - \frac{1}{4}R = 0 \tag{3}$$

where ρ and λ are defined below

$$\rho = \left[\frac{8\mu(-E)}{\hbar^2}\right]^{1/2} r \qquad \lambda = \left[\frac{Z^2 e^4 \mu}{2\hbar^2(-E)}\right]^{1/2}. \tag{4}$$

$Y_{l,m}(\theta, \phi)$ are the familiar spherical harmonics which are simultaneous eigenfunctions of L^2 and L_z. By the single-valuedness requirement, the quantum number m must be integral. Furthermore, the integral valuedness of the quantum number l is required for the consistency of group representation [5]. In fact, in the case of hydrogen-like atoms, the quantum number l must be integral if we want to obtain the same energy spectrum as the Bohr model. When we seek solutions for $R(r)$ in the form

$$R(r) = \exp(-\rho/2)\rho^l S(\rho) \tag{5}$$

then, by substitution into equation (3), we obtain the following differential equation for the function $S(\rho)$:

$$\frac{\mathrm{d}^2 S}{\mathrm{d}\rho^2} + \left(\frac{2l+1}{\rho} - 1\right)\frac{\mathrm{d}S}{\mathrm{d}\rho} + \frac{\lambda - l - 1}{\rho}S = 0. \tag{6}$$

This equation can be solved by a series expansion of $S(\rho)$

$$S(\rho) = \sum_{n=0}^{\infty} a_n \rho^n \tag{7}$$

with the coefficients a_n satisfying the recursion relation

$$a_{n+1} = \frac{n + l + 1 - \lambda}{(n+1)(n+2l+2)} a_n. \tag{8}$$

This result and relation (4) show that the quantum number l must be integral for the energy E to have the same form as that of the Bohr model.

Now let us examine the hydrogen-like atom with the viewpoint of an observer who sees it as a planar system. Since the Schrödinger equation of form (1) is invariant under rotations, this observer can still invoke the Schrödinger equation for an analysis of the dynamics of the atomic system. However, in this case there are two important aspects which relate to the topology of the system which we must emphasize and exploit in our discussion. First, the configuration space of the atom is now multiply connected and, as mentioned above, multivalued wavefunctions can be used. Second, in two-dimensional space, the more suitable coordinate system is the planar polar coordinates and so the Schrödinger equation now takes the form

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\phi)-\frac{Ze^2}{r}\psi(r,\phi)=E\psi(r,\phi). \tag{9}$$

Here we have assumed that the Coulomb potential retains its form. Solutions of the form $\psi(r,\phi) = R(r)\Phi(\phi)$ then reduce the above equation to two separate equations for the functions Φ and R

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}+m^2\Phi=0 \tag{10}$$

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r}-\frac{m^2}{r^2}R+\frac{2\mu}{\hbar^2}\left(\frac{Ze^2}{r}-E\right)R=0 \tag{11}$$

where m is identified as the angular momentum of the system. From equation (10), we obtain a solution for the function Φ in the form

$$\Phi(\phi)=\exp(\mathrm{i}m\phi). \tag{12}$$

With this form of solutions, we normally require the angular momentum m to take integral values so that the single-valuedness condition is satisfied. However, we will show that the integral-valuedness requirement for the quantity m in this case is not compatible with the assumption that an observer in two-dimensional space should also obtain a spectrum of energy like that of the Bohr model. It should be emphasized here that the Bohr model is also a planar system.

From equation (11), if we define ρ and λ as in (4) then we can write this equation in a simpler form

$$\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho}-\frac{m^2}{\rho^2}R+\frac{\lambda}{\rho}R-\frac{1}{4}R=0. \tag{13}$$

We notice that the crucial difference between the equation for R, in this case, and equation (3) is the factor 2 before $\mathrm{d}R/\mathrm{d}r$. This equation can also be solved by writing R in the form (5) as before. The equation for the function $S(\rho)$ now differs slightly from equation (6) by the coefficient before S:

$$\frac{\mathrm{d}^2S}{\mathrm{d}\rho^2}+\left(\frac{2m+1}{\rho}-1\right)\frac{\mathrm{d}S}{\mathrm{d}\rho}+\left(\frac{\lambda-m-\frac{1}{2}}{\rho}\right)S=0. \tag{14}$$

A power-series solution (7) for the function S will result in the following recursion relation:

$$a_{n+1}=\frac{n+m+\frac{1}{2}-\lambda}{(n+1)(n+2m+1)}a_n. \tag{15}$$

With this result and by relation (4), the energy spectrum in this case can be written explicitly in the form

$$E = -\frac{Z^2e^4\mu}{2\hbar^2(n+m+\frac{1}{2})^2}. \tag{16}$$

Hence, if the hydrogen-like atom is viewed as a two-dimensional physical system and if the energy is observed to have the same spectrum as that of the Bohr model, then the angular momentum m must take half-integral values. However, it can be verified that integral values for the angular momentum m can be retained if we add to the Coulomb potential a quantity $-(\hbar\sqrt{E/2\mu})/r$ when the hydrogen-like atom is viewed as a two-dimensional physical system.

We have shown in a simple but explicit way that the topological structure of a configuration space of a physical system can determine the quantum nature of an observable of the system. This result should be expected in quantum mechanics since we know that *the quantum behaviour of a particle depends almost entirely on the configuration of an* experiment. If, in a particular experiment, the electron of a hydrogen-like atom is constrained to move in a plane, then the orbital angular momentum of the electron must take half-integral values if we use the Schrödinger equation to study the dynamics of the electron and want to retain the same energy spectrum as the Bohr model. As a consequence, it might seem possible to invoke the result to explain the Stern–Gerlach experiment without the necessity of introduction of spin into the quantum theory. In this case, there emerges a problem that, *in order to incorporate the situation into the dynamics of the electron in three-dimensional* space, we would have to describe the physical system of the hydrogen-like atom in terms of the mathematics of its possible topological structures.

Let us speculate on another possible related problem that arises from the multiply connectedness of a configuration space of a physical system and the multivaluedness of wavefunctions in this space. In physics, singular behaviours of a physical quantity are sometimes due to a breakdown of the coordinates used for its description, and, in such cases, singularities can be removed by suitable coordinate transformations. Such a singularity is the well known Schwarzschild singularity at $r = 2m$ in general relativity. In Einstein's theory of general relativity, the Schwarzschild metric takes the following form [10]:

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{17}$$

With a solution in this form, the metric components become singular at both $r = 2m$ and $r = 0$. However, it has been shown that, while the singularity at $r = 0$ is a true physical singularity, the singularity at $r = 2m$ is only a coordinate breakdown [11, 12]. However, we can ask the interesting question as to whether the Schwarzschild singularity at $r = 2m$ is real and detectable with respect to an observer who uses the Schwarzschild coordinates. This problem may be seen as an analogue to the problem discussed in this paper in the sense that if we consider the planar configuration of the hydrogen-like atom simply connected, then the wavefunctions must be non-singular and the angular momentum must take integral values. However, when the topological structure of the physical system of the hydrogen-like atom is multiply connected, the wavefunctions can be multivalued, and then the angular momentum may take half-integral values. The singularity of the wavefunctions of the Schrödinger wave equation in quantum mechanics, in this case, can, therefore, be regarded as similar to the Schwarzschild singularity of the Schwarzschild metric, which is a solution of the Einstein field equations of general relativity.

Acknowledgments

I would like to thank a referee for constructive comments and Dr Michael J Morgan for useful discussions on topological methods in physics. I also acknowledge the financial support of an APA Research Award.

References

[1] Feynman R P 1948 *Rev. Mod. Phys.* **20** 367–87
[2] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[3] Yourgrau W and Mandelstam S 1968 *Variational Principles in Dynamics and Quantum Theory* (New York: Dover)
[4] Dittrich W and Reuter M 1992 *Classical and Quantum Dynamics* (Berlin: Springer)
[5] Merzbacher E 1962 *Am. J. Phys.* **30** 237–43
[6] Morandi G 1992 *The Role of Topology in Classical and Quantum Physics* (Berlin: Springer)
[7] Balachandran A P, Marmo G, Skagerstam B S and Stern A 1991 *Classical Topology and Quantum States* (Singapore: World Scientific)
[8] Landau L D and Lifshitz E M 1987 *Quantum Mechanics* (Sydney: Pergamon)
[9] Bransden B H and Joachan C J 1989 *Introduction to Quantum Mechanics* (New York: Longman)
[10] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (New York: Freeman)
[11] Kruskal M D 1960 *Phys. Rev.* **119** 1743–5
[12] Wald R M 1984 *General Relativity* (Chicago: Chicago University Press)